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New similarity reductions of the steady-state boundary layer equations

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Abstract. New similarity reductions and exact solutions of the steady-state boundary layer equations, which are of considerable importance in physics, are presented. Besides similarity reductions obtained by applying the 'non-classical' group method due to Bluman and Cole, some similarity solutions, which figure neither among classical nor among non-classical reductions, are determined using an extension of the non-classical method.

1. Introduction

The classical method for finding similarity reductions of partial differential equations (PDE) is the Lie-group method of infinitesimal transformations (see, for example, Bluman and Cole (1974) and Olver (1986)). To apply the method to the nth-order equation for a function $u(x,t)$ of two variables of the form

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},\ldots)=0 \tag{1.1}$$

where the subscripts on u denote partial derivatives, we consider the one-parameter (ϵ) Lie group of infinitesimal transformations in (x,t,u) given by

$$x^\star = x+\epsilon\xi(x,t,u)+\mathrm{O}(\epsilon^2) \tag{1.2a}$$
$$t^\star = t+\epsilon\tau(x,t,u)+\mathrm{O}(\epsilon^2) \tag{1.2b}$$
$$u^\star = u+\epsilon\eta(x,t,u)+\mathrm{O}(\epsilon^2) \tag{1.2c}$$

which leaves (1.1) invariant.

The vector fields of the group generators ($\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$) are determined from the invariance requirement

$$pr^{(n)}\Delta|_{\Delta=0}=0 \tag{1.3}$$

where $pr^{(n)}$ indicates the nth prolongation of the transformations group. Having defined the generators of a symmetry group from the invariant surface condition

$$\Delta^{(1)}=\xi(x,t,u)u_x+\tau(x,t,u)u_t-\eta(x,t,u)=0 \tag{1.4}$$

the similarity reductions may be obtained after determination of the group invariants using the general integral of the characteristic system. The original equation (1.1) is rewritten in terms of group invariants (treating one of them, $w(x,t,u)$, as a function of the other invariant $z(x,t,u)$) and thus is reduced to an ordinary differential equation (ODE) for $w(z)$.

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Bluman and Cole (1969) proposed a generalization of Lie's method for finding group-invariant solutions which they named the 'non-classical' method. In this approach the condition for the invariance of the PDE is replaced by weaker conditions for the invariance of the combined system of differential equations consisting of the original differential equation along with equation (1.4). In other words we require that the appropriate prolongation of the vector field should annihilate both equations on the solution surface of both equations:

$$pr^{(n)}\Delta|_{\Delta=0,\Delta^{(1)}=0} = 0 \tag{1.5}$$

$$pr^{(1)}\Delta^{(1)}|_{\Delta=0,\Delta^{(1)}=0} = 0. \tag{1.6}$$

This situation is subsumed under the general concept of a differential equation with side conditions proposed by Olver and Rosenau (1986). Note that equation (1.6) is satisfied trivially and is hence no restriction on the group generators. The relation (1.5) gives rise to the differential system of equations for group generators which, as distinct from the system produced by the classical Lie group method, is in general nonlinear. The set of solutions for the generators potentially available with the help of this method is larger than the set obtained by the classical method, making it possible to find further types of explicit solutions by the same reduction techniques.

A generalization of the Bluman–Cole non-classical method was proposed by Olver and Rosenau (1987). To find the similarity solutions of a given PDE, Olver and Rosenau (1987) start from any group G of point transformations and proceed to apply the usual approach to construct group-invariant solutions. Since in general G is not a symmetry group of the original PDE, the reduction procedure results in an equation which involves, besides invariants of the group and their derivatives, one of the independent variables (for example x) as a parametric variable. This equation can be reduced to an overdetermined system of ODEs (these can be found, for instance, by expanding the equation in powers of x). Generally, the system will not be compatible for a specific group and equation, and therefore the validity of the method is restricted by the impossibility of determining *a priori* which groups will result in compatible reduced systems.

The 'direct method' of Clarkson and Kruskal (1989) does not use group theory. The basic idea of this method is to seek the solution of equation (1.1) in the form

$$u(x,t) = F(x,t,w(z(x,t))) \tag{1.7}$$

which could be considered as the most general form for similarity solutions (see Bluman and Cole (1974)). Substituting the ansatz (1.7) into (1.1) and requiring that the result be an ODE for $w(z)$ imposes conditions upon F, z and their derivatives, which yields the desired reductions. Note that for most of the equations to which the method was applied, it turned out to be sufficient to use a special form of (1.7), namely

$$u = \alpha(x,t) + \beta(x,t)w(z(x,t)). \tag{1.8}$$

An extended version of the direct method of Clarkson and Kruskal is introduced in Hood (1995). This new method is similar to the original but begins with a generalization of the form (1.8), namely

$$u = \alpha(x,t) + \beta(x,t)w(z(x,t)) + \gamma(x,t)v(\zeta(x,t)) \tag{1.9}$$

and correspondingly uses a more general concept, which is to seek reductions to a system of ordinary differential equations rather than the usual single equation; this leads to a wider class of solutions.

Relations existing between the non-classical method due to Bluman and Cole (1969) and the direct method due to Clarkson and Kruskal (1989) were discussed in a series of papers concerning applications of the direct method—see, for example, Levi and Winternitz (1989), Nucci and Clarkson (1992), Pucci (1992), Clarkson (1992), Arrigo *et al* (1993) and Olver (1994). For some equations (e.g. the Boussinesq equation; see Levi and Winternitz (1989)) it was established that the solutions given by the Clarkson and Kruskal direct reduction procedure are exactly the same as those obtained as invariant solutions under the non-classical symmetry groups admitted by the equation. However, other equations (Nucci and Clarkson 1992, Pucci 1992) appear to indicate that the non-classical method is more general than the direct method. It was observed by Pucci (1992) that the similarity solutions corresponding to non-classical groups should, in general, constitute a larger family than that obtained by the Clarkson and Kruskal method, since the reduction (1.7) used in the direct method is equivalent to finding similarity solutions corresponding only to those non-classical groups for which the ratio ξ/τ of generators is independent of u and $\tau \neq 0$. Pucci's results have been somewhat extended by Arrigo *et al* (1993). Olver (1994) has shown that the direct method is equivalent to the non-classical method when the generators for the independent variables are autonomous with respect to the dependent variables.

Another method for deriving special explicit solutions of PDEs was developed in Burde (1990, 1994) and applied to the axisymmetric steady-state boundary-layer equations. This method involves the usual direct substitution of a similarity form in a given PDE, but in contrast to other methods it does away with the usual requirement of reducibility to an ODE. Instead, the PDE is reduced to an overdetermined system of ODEs for $w(z)$ that can be solved in closed form (similar ideas have been used by Galaktionov (1990) to construct an exact solution of the nonlinear heat equation with a source term and by Estevez (1992) to obtain a particular solution of the Fitzhugh–Nagumo equation). Some arguments presented in Burde (1994) suggest that the similarity reductions produced by this method can be obtained neither by the group methods (classical or non-classical) nor by the direct method of Clarkson and Kruskal (1989). A generalization of the method to the case of a PDE with three independent variables is given in Burde (1995).

In this paper we discuss similarity reductions of the two-dimensional steady-state boundary layer (BL) equations. The classical symmetry groups for these equations are determined in Ovsiannikov (1982). We apply the non-classical method due to Bluman and Cole (1969) to the steady-state BL equations reduced to a single equation for a stream function

$$u_{xxx} + u_t u_{xx} - u_x u_{xt} - \Theta = 0 \tag{1.10}$$

where x and t are respectively the transverse and longitudinal Cartesian coordinates and $\Theta(t)$ is an arbitrary element. We show that all similarity reductions corresponding to non-classical symmetries of the Bluman and Cole type (1969) (for which $\tau \neq 0$) can be found for this equation. Further we develop an extension of the non-classical method which provides a unifying group-theoretic framework for the Clarkson and Kruskal (1989) method and the method developed in Burde (1990, 1994), and show that applied to equation (1.10) this extension can produce new similarity reductions not found among classical and non-classical similarity reductions.

The outline of the paper is as follows. In section 2 we describe the classical similarity reductions of the BL equations and apply the non-classical method due to Bluman and Cole (1969) to equation (1.10). In section 3 we develop the extension to the non-classical method and apply it to equation (1.10) to derive some similarity solutions not obtainable with the non-classical method. Finally, in section 4 we make some remarks on the results and suggest possible further work. Application of the direct method of Clarkson and Kruskal to equation (1.10) is outlined in appendix A. Auxiliary results concerning some generalization of our method are presented in appendixes B and C.

2. Classical and non-classical similarity reductions of the BL equations

2.1. Boundary layer equations

Consider a steady two-dimensional viscous flow of an incompressible fluid over a flat plate with the latter taken as $Y = 0$ in the Cartesian (X, Y) coordinates. The corresponding components of velocity are denoted respectively by U and V. For a high Reynolds number, the flow is described by the BL equations (see Schlihting (1968))

$$U_X + V_Y = 0 \tag{2.1a}$$

$$UU_X + VU_Y = U^{(e)}U_X^{(e)} + U_{YY} \tag{2.1b}$$

where $U^{(e)}(X)$ is a given external flow. Without loss of generality we set the fluid kinematic viscosity equal to one, which amounts to choosing suitable units for length and time. Physically significant solutions of the BL equations should satisfy the condition at infinity

$$U(X, Y) = U^{(e)}(X) \qquad \text{as} \quad Y \to \infty. \tag{2.2}$$

The boundary conditions at $Y = 0$ may have different forms corresponding to different physical conditions at the body surface (besides the usual zero conditions for U and V, specific laws of suction or blowing could be prescribed, for example).

In what follows we will deal with the BL equations reduced to one equation for the stream function Ψ defined by $U = \Psi_Y$ and $V = -\Psi_X$. We will also change the notation: $\{\Psi, Y, X\} \to \{u, x, t\}$ and $U^{(e)}U_X^{(e)} \to -\Theta(t)$ to make it similar to that used in group theoretic considerations. Then the equation for the stream function takes the form (1.10) which is used in all subsequent calculations.

2.2. Classical similarity reductions

Equation (1.10) admits the classical symmetry groups defined by the following generators (see Ovsiannikov 1982):

$$\xi = C_3x + \phi(t) \qquad \tau = (2C_3 + C_2)t + C_1 \qquad \eta = (C_3 + C_2)u + C_4 \tag{2.3}$$

with C_1, C_2, C_3 and C_4 being arbitrary constants. The constants C_1, C_2 and C_3 are connected with the equation for the arbitrary element Θ:

$$[C_1 + (C_2 + 2C_3)t]\Theta_t + (2C_3 - C_2)\Theta = 0. \tag{2.4}$$

The constant C_4 is insignificant: any constant could be included in the stream function u without loss of generality as it does not effect changes in the velocity components. The notation of the constants in (2.3) and (2.4) coincides with that in Ovsiannikov (1982) but to compare the generators one should go over from the stream function to the velocity components used as dependent variables in Ovsiannikov (1982).

All classical similarity reductions can be obtained by solving the characteristic equations to find the invariants of the group, which are then used as new variables. For our case, the characteristic equations are

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}u}{\eta} \tag{2.5}$$

where ξ, τ and η are given in (2.3).

Let us first consider the particular case when $C_3 \neq 0$, $C_3 \neq -C_2$ and $C_3 \neq -C_2/2$. With suitable rescaling and renaming of the parameters C_1, C_2, C_3 and $\phi(t)$, (2.5) takes the form

$$\frac{\mathrm{d}x}{A(x+\Phi(t))} = \frac{\mathrm{d}t}{(A+1)(t-t_0)} = \frac{\mathrm{d}u}{u} \tag{2.6}$$

where $A = C_3/(C_3+C_2)$, $t_0 = -C_1/(2C_3+C_2)$ and $\Phi = \phi/C_3$. Unless otherwise stated, we shall set $t_0 = 0$. The invariants $z(x,t)$ and $w(t,u)$ are determined as the integration constants of the equations (2.6) as follows

$$z = t^{-n}x + q(t) \qquad w = ut^{n-1} \tag{2.7}$$

where $n = A/(A+1)$ is an arbitrary constant and $q(t) = -n\int(\Phi(t)/t^{n+1})\,\mathrm{d}t$ is an arbitrary function. Treating one of the invariants w as a function of the other invariant z, one obtains the similarity reduction

$$u = t^{1-n}w(z) \qquad z = t^{-n}x + q(t) \tag{2.8}$$

where n is an arbitrary constant and $q(t)$ is an arbitrary function. Equation (2.4) is easily solved for Θ to give

$$\Theta = \Theta_0 t^{1-4n} \tag{2.9}$$

where Θ_0 is a constant.

Similarity reductions obtained in other particular cases have the following forms

$$C_3 = 0: \quad u = tw(z) \qquad z = x + q(t) \qquad \Theta = \Theta_0 t \tag{2.10}$$

$$C_3 = -C_2: \quad u = w(z) \qquad z = x/t + q(t) \qquad \Theta = \Theta_0 t^{-3} \tag{2.11}$$

$$C_3 = -C_2/2: \quad u = \mathrm{e}^{\lambda t}w(z) \qquad z = x\,\mathrm{e}^{\lambda t} + q(t) \qquad \Theta = \Theta_0\,\mathrm{e}^{4\lambda t} \tag{2.12}$$

where λ is a constant and $q(t)$ is an arbitrary function. Note that the reductions (2.10) and (2.11) are particular cases of the reduction (2.8).

2.3. Non-classical similarity reductions

Applying the non-classical method due to Bluman and Cole (1969) to equation (1.10) and assuming $\tau \neq 0$ (we set $\tau = 1$ without loss of generality) we obtain the following equations for the group generators:

$$\xi_u = 0 \qquad \eta_{uu} = 0 \tag{2.13}$$

$$3\xi_{xx} - 3\eta_{xu} - \eta_t - \xi\eta_x - \eta\eta_u - \eta\xi_x = 0 \tag{2.14}$$

$$\eta_{tu} - \xi_{xt} + \xi\eta_{xu} - \xi\xi_{xx} + \eta_u^2 - \xi_x^2 = 0 \tag{2.15}$$

$$\xi_{xxx} - 3\eta_{xxu} + \eta_{xt} + \xi\eta_{xx} - \eta\eta_{xu} + \eta\xi_{xx} + 2\eta_x\eta_u = 0 \tag{2.16}$$

$$\Theta_t - \Theta(\eta_u - 3\xi_x) - \eta_{xxx} - \eta\eta_{xx} + \eta_x^2 = 0. \tag{2.17}$$

We do not consider the case $\tau = 0$ restricting ourselves to non-classical symmetries of the Bluman and Cole type (1969).

From (2.13) one deduces that

$$\eta = M(x,t) + N(x,t)u \qquad \xi = L(x,t). \tag{2.18}$$

Substituting (2.18) into (2.14)–(2.16) yields a system of equations for the functions $M(x,t)$, $N(x,t)$ and $L(x,t)$ from which one can see by a direct check that this system has no solutions with $N_x \neq 0$. Then there are four cases to consider for $\tau \neq 0$ and correspondingly the following four families of non-classical group generators are defined:

Case 1: $N_x = 0$, $N_t \neq 0$, $M_x \neq 0$.

$$\tau = 1 \qquad \xi = \tfrac{1}{3}x(t-t_0)^{-1} + \phi(t) \tag{2.19a}$$

$$\eta = \tfrac{2}{3}u(t-t_0)^{-1} + C_1 x(t-t_0)^{-4/3} - C_1(t-t_0)^{-1}\int (t-t_0)^{-1/3}\phi(t)\,\mathrm{d}t + C_2(t-t_0)^{-1} \tag{2.19b}$$

$$\Theta = \tfrac{3}{4}C_1^2(t-t_0)^{-5/3} + \Theta_0(t-t_0)^{-1/3}. \tag{2.19c}$$

Case 2: $N_x = 0$, $N_t \neq 0$, $M_x = 0$.

$$\tau = 1 \qquad \xi = (1-C_1)x(t-t_0)^{-1} + \phi(t) \tag{2.20a}$$

$$\eta = (C_1 u + C_2)(t-t_0)^{-1} \qquad \Theta = \Theta_0(t-t_0)^{4C_1-3}. \tag{2.20b}$$

Case 3: $N = \text{constant} \neq 0$.

$$\tau = 1 \qquad \xi = -C_1 x + \phi(t) \qquad \eta = C_1 u + C_2 \qquad \Theta = \Theta_0\,\mathrm{e}^{4C_1 t}. \tag{2.21}$$

Case 4: $N = 0$.

$$\tau = 1 \qquad \xi = x(t-t_0)^{-1} + \phi(t) \qquad \eta = C_2(t-t_0)^{-1} \qquad \Theta = \Theta_0(t-t_0)^{-3}. \tag{2.22}$$

In all the equations (2.19)–(2.22), C_1, C_2, t_0 and Θ_0 are arbitrary constants and $\phi(t)$ is an arbitrary function.

Solving the characteristic equations (2.5) we obtain the corresponding similarity reductions (we rescale and rename arbitrary constants and functions and set $t_0 = 0$):

Case 1.

$$u = cz + t^{2/3}w(z) \qquad z = xt^{-1/3} + q(t) \qquad \Theta = \Theta_0 t^{-1/3} + \tfrac{1}{3}c^2 t^{-5/3} \tag{2.23a}$$

where c is an arbitrary constant, $q(t)$ is an arbitrary function and $w(z)$ satisfies

$$w''' + \tfrac{2}{3}ww'' - \tfrac{1}{3}(w')^2 - \Theta_0 = 0. \tag{2.23b}$$

Hereafter primes denote derivatives with respect to an argument in any function of one variable.

Case 2i: $C_1 \neq 0$.

$$u = t^{1-n}w(z) \qquad z = xt^{-n} + q(t) \qquad \Theta = \Theta_0 t^{1-4n} \tag{2.24a}$$

where $n = 1 - C_1$ is an arbitrary constant, $q(t)$ is an arbitrary function and $w(z)$ satisfies

$$w''' + (1-n)ww'' + (2n-1)(w')^2 - \Theta_0 = 0. \tag{2.24b}$$

Case 2ii: $C_1 = 0$.

$$u = k\ln t + w(z) \qquad z = xt^{-1} + q(t) \qquad \Theta = \Theta_0 t^{-3} \tag{2.25a}$$

where k is an arbitrary constant, $q(t)$ is an arbitrary function and $w(z)$ satisfies

$$w''' + kw'' + (w')^2 - \Theta_0 = 0. \tag{2.25b}$$

Case 3.

$$u = e^{\lambda t} w(z) \qquad z = x\, e^{\lambda t} + q(t) \qquad \Theta = \Theta_0\, e^{4\lambda t} \tag{2.26a}$$

where $\lambda = C_1$ is an arbitrary constant and $q(t)$ is an arbitrary function and $w(z)$ satisfies

$$w''' + \lambda w w'' - 2\lambda (w')^2 - \Theta_0 = 0. \tag{2.26b}$$

Case 4. It is easily seen that the set of infinitesimals (2.22) is a particular case of the set given by (2.20) for $C_1 = 0$. Hence, the similarity reduction in case 4 is given by (2.25).

It should be noted that in all the formulae (2.23)–(2.26) insignificant additive constants have been omitted in the expression for u.

Among the similarity reductions found above by applying the non-classical method due to Bluman and Cole, only the reduction (2.23) is a new one. Reductions (2.24) and (2.26) coincide with the classical similarity reductions (2.8) and (2.12). Reduction (2.25) has been identified in Burde (1990) using a solution form which is formally a subset of ansatz (1.8).

3. Some other reductions—an extension to the non-classical method

To formulate the extension we will make some changes in the procedure of the non-classical method.

Both the original procedure of the non-classical method due to Bluman and Cole (1969) and our procedure start from the infinitesimal equation (1.3) obtained by the prolongation of the group action on (1.1). In the procedure of the non-classical method the invariant surface condition (1.4) and its differential consequences up to order n are used to eliminate derivatives u_t, u_{xt} and so on from (1.3) (some difficulties encountered in this process and the ways to avoid them are discussed by Clarkson and Mansfield (1994)). As a result this infinitesimal equation is represented as a polynomial in the derivatives u_x, u_{xx} and so on. Then setting the coefficients of the monomials to zero yields the determining equations for the group generators. Having the generators defined, the invariant surface condition (1.4) allows determination of the similarity variable and the form of similarity reduction.

In our procedure a specific form of similarity reduction is assumed from the beginning and the conditions for the independent z and dependent w similarity variables, being group invariant, are used as auxiliary equations instead of the invariant surface condition (1.4). These auxiliary equations enable one to express the group generators as functions of x, t and u and eliminate them from (1.3); next the assumed similarity form is used for eliminating u from the equation obtained. The transformations are aimed at representing the resulting infinitesimal equation as a polynomial in $w(z)$ and its derivatives (instead of a polynomial in the derivatives of u with respect to x as in the original procedure of the non-classical method). Satisfying this equation determines parameters of the assumed similarity form.

Let us define this procedure more concretely assuming the similarity form (1.8). In this case the similarity variables are represented as functions of x, t and u by the following

$$z = z(x,t) \qquad w = \zeta(x,t,u) = \frac{u}{\beta(x,t)} - \frac{\alpha(x,t)}{\beta(x,t)}. \tag{3.1}$$

The conditions for the invariance of $z(x,t)$ and $\zeta(x,t,u)$ under group transformations are

$$I_1 = \xi z_x + \tau z_t = 0 \tag{3.2a}$$

$$I_2 = \xi \zeta_x + \tau \zeta_t + \eta \zeta_u = 0. \tag{3.2b}$$

The group generators $\xi(x,t)$ and $\eta(x,t,u)$ (we set $\tau = 1$) are expressed from (3.2) as follows

$$\xi = -\frac{z_t}{z_x} \tag{3.3}$$

$$\eta = u\left(\frac{\beta_t - (z_t/z_x)\beta_x}{\beta}\right) + \alpha_t - (z_t/z_x)\alpha_x - \frac{\alpha}{\beta}(\beta_t - (z_t/z_x)\beta_x) \tag{3.4}$$

which enables the derivatives of the generators with respect to x, t and u to be calculated and, in this way, the generators are eliminated from equation (1.3).

After eliminating u with the use of (1.8) and its differential consequences:

$$u_x = \alpha_x + \beta_x w + \beta z_x w' \qquad u_t = \alpha_t + \beta_t w + \beta z_t w' \qquad \ldots \tag{3.5}$$

up to order n, the resulting infinitesimal equation is rearranged by collecting coefficients of like derivatives and powers of $w(z)$.

The described procedure is equivalent to the non-classical method in a sense, since both (1.4), used as an auxiliary equation in the original procedure of the method, and (3.2*a*), (3.2*b*) and (1.8), used as auxiliary equations in our procedure, prescribe the group invariance of the desired solutions (it is easily checked that (1.4) is satisfied by (3.3), (3.4) and (1.8)). At the same time the form in which the resulting infinitesimal equation is represented permits an extension as shown below.

We will show this resulting equation for the case when the original PDE is (1.10). Then the infinitesimal equation (1.3), with which the procedure starts, has the form

$$\begin{aligned} P = {} & u_{xx}u_t(\xi_x + \eta_u) + u_{xt}u_x(-\xi_x - \eta_u) + u_{xx}(-3\xi_{xx} + 3\eta_{xu} + \eta_t) + u_{xt}(-\eta_x) \\ & + u_x^2(\xi_{xt} - \eta_{tu}) + u_x u_t(-\xi_{xx} + \eta_{xu}) + u_x(3\eta_{xxu} - \xi_{xxx} - \eta_{xt}) + u_t(\eta_{xx}) \\ & + [\eta_{xxx} - \Theta' - \Theta(3\xi_x - \eta_u)] = 0. \end{aligned} \tag{3.6}$$

To avoid unnecessary complications, we have already taken into account that $\tau = 1$, $\xi_u = 0$ and $\eta_{uu} = 0$, as follows from (3.3) and (3.4). The resulting infinitesimal equation takes the following form:

$$\begin{aligned} P = {} & \beta z_x^3[B^{(1)}ww'' + B^{(2)}(w')^2 + B^{(3)}ww' + B^{(4)}w^2 + B^{(5)}w'' + B^{(6)}w' + B^{(7)}w + B^{(8)}] \\ & = 0 \end{aligned} \tag{3.7}$$

where the coefficients $B^{(i)}(x,t)$ are expressed through the functions $\alpha(x,t)$, $\beta(x,t)$ and $z(x,t)$.

The combinations appearing in these expressions (similar combinations can be seen in (3.4)) suggest the following change of variables:

$$\{x,t\} \to \{z(x,t), \varphi\} \qquad \varphi = t \tag{3.8a}$$

so that

$$f_x = f_z z_x \qquad f_t = f_\varphi + f_z z_t \qquad f_t - (z_t/z_x)f_x = f_\varphi \tag{3.8b}$$

for any function $f(x,t)$ (we retain the same notation for the functions of new variables). Then the expressions for the coefficients $B^{(i)}(z,\varphi)$ take the following forms:

$$B^{(1)} = \frac{\beta}{\gamma}(b_{\varphi\varphi} + b_\varphi^2 - b_\varphi g_\varphi) \qquad B^{(2)} = \frac{\beta}{\gamma}(-b_{\varphi\varphi} - b_\varphi^2 + g_\varphi^2 - g_{\varphi\varphi}) \qquad \ldots \tag{3.9a}$$

where

$$\gamma(z,\varphi) = z_x \qquad g(z,\varphi) = \ln\gamma \qquad b(z,\varphi) = \ln\beta \tag{3.9b}$$

and the following identities have been used:

$$\left(\frac{z_t}{z_x}\right)_x = g_\varphi \qquad \beta_\varphi = \beta b_\varphi. \tag{3.9c}$$

It turns out that there exists a close connection between the resulting infinitesimal equation (3.7) of our procedure and the initial relation of the direct method. The latter relation is obtained by substituting the assumed similarity form (1.8) and its differential consequences (3.5) into the PDE (1.10). After collecting coefficients of like derivatives and powers of $w(z)$, normalizing all the coefficients by dividing them by the coefficient of the monomial w''' and making the change of variables (3.8) in the normalized coefficients (for more details see appendix A), the initial relation of the direct method takes the form (A3) with the normalized coefficients $A^{(i)}(z,\varphi)$ given by (A4).

The connection between (3.7) and (A3) displays itself in relations existing between the coefficients $B^{(i)}(z,\varphi)$ of the monomials in (3.7) and the corresponding normalized coefficients $A^{(i)}(z,\varphi)$ in (A3), namely

$$B^{(i)} = A^{(i)}_\varphi \qquad i = 1, 2, \ldots, 8. \tag{3.10}$$

One may directly check these relations for $i = 1$ and $i = 2$ differentiating (A4*a*) with respect to φ with allowance for (3.9*c*) and comparing the results with (3.9*a*). The representation (3.10) of the coefficients in equation (3.7), which is the condition for the group invariance of the PDE (1.10) supplemented by the auxiliary equations prescribing group invariance of the desired solutions, on the one hand establishes a direct connection of the non-classical method with the direct method and on the other hand leads to an extension of the non-classical method.

First, if one satisfies equation (3.7) requiring that all the coefficients $B^{(i)}$ of monomials in (3.7) vanish, this with allowance for (3.10) leads to the direct method. Indeed, the requirement $A^{(i)}_\varphi = 0$ is equivalent to the basic requirement of the direct method of Clarkson and Kruskal (1989) that the ratios of the coefficients of different monomials in the initial relation of the direct method be functions of z exclusively.

Further, one may satisfy (3.7) in another way, namely to require that it is reduced to an ODE for the function $w(z)$. This implies that the ratios of the coefficients $B^{(i)}(z,\varphi)$ of different monomials in (3.7) are functions of z which yields the following set of equations (we assume for definiteness that $B^{(1)} \neq 0$):

$$A^{(i)}_\varphi = Q_i(z)A^{(1)}_\varphi \qquad i = 2, 3, \ldots, 8 \tag{3.11}$$

$$ww'' + Q_2(w')^2 + Q_3ww' + Q_4w^2 + Q_5w'' + Q_6w' + Q_7w + Q_8 = 0. \tag{3.12}$$

Equations (3.11) can be easily integrated to give

$$A^{(i)}(z,\varphi) = Q_i(z)A^{(1)}(z,\varphi) + K_i(z) \qquad i = 2, 3, \ldots, 8 \tag{3.13}$$

where $K_i(z)$ are functions of integration. The relations (3.13) coupled with the expressions (A4) for $A^{(i)}$ yield differential equations for the functions $\alpha(z,\varphi)$, $\beta(z,\varphi)$ and $\gamma(z,\varphi) = z_x$ which specify the form of the similarity reduction. To find $w(z)$ one has to substitute in the usual way this form into the original PDE (1.10) or to substitute the relations (3.13) into equation (A3) derived from (1.10) and (1.8). It is easily seen that the substitution of (3.13) into (A3) yields an overdetermined system of two ODEs for $w(z)$ one of which is (3.12) and the other is

$$w''' + K_2(w')^2 + K_3ww' + K_4w^2 + K_5w'' + K_6w' + K_7w + K_8 = 0. \tag{3.14}$$

Thus, the desired similarity solution is defined by (1.8) with $\alpha(x,t)$, $\beta(x,t)$, $z(x,t)$ and $w(z)$ determined from the system (3.13), (3.12) and (3.14) where $Q_i(z)$ and $K_i(z)$ are to be partially or completely specified in the process of solving the system. Evidently this method should, in general, yield reductions differing from those obtained by the non-classical method.

The relations (3.13) are similar to those used in the method developed in Burde (1990, 1994) when two of the coefficients of monomials in the relation (A3) serve as the normalizing coefficients to reduce (A3) to an overdetermined system of two ODEs for $w(z)$ (here these normalizing coefficients are the coefficients of the monomials w''' and ww''). The procedure described above not only provides a group-theoretic explanation of the method but also gives an algorithm for constructing the system of ODEs for $w(z)$. The technique for applying the method is outlined below.

3.1. Technique

To apply the method, one assumes a specific form for a similarity reduction (the form (1.8) is not obligatory—for example, below we show an application of the method using a form which is a special case of (1.8)). The assumed form is substituted into the PDE and the relation obtained is rearranged by collecting like derivatives and powers of $w(z)$. The coefficients of the monomials are normalized by dividing by one of the coefficients. If the highest (nth-order) derivative of u with respect to x enters the original PDE linearly, which is frequently the case, the coefficient of the nth-order derivative of $w(z)$ can always be taken as the normalizing one (for ease of exposition we will assume this below). Let us also assume that the remaining monomials and their coefficients $A^{(i)}$ are indexed from 1 to M. The relation obtained in this way is a counterpart of (A3); it is also the initial relation of the direct method and all the procedure up to this point coincides with that of the direct method.

Next one makes the change of variables (3.8) in the expressions for the normalized coefficients $A^{(i)}$, after which the system of equations defining parameters of the assumed similarity form can be constructed. This system consists of two ODEs for $w(z)$ and the system of PDEs for other functions of z and φ entering the similarity form.

One of the ODEs for $w(z)$ (a counterpart of equation (3.12)) does not include the nth-order derivative of $w(z)$; the monomials, for which $A_\varphi^{(i)} = 0$, are also not included. The coefficients $Q_i(z)$ of the monomials in this ODE are functions to be determined and one of the coefficients, let us say, with index N (in (3.12) $N = 1$), is taken to be equal to one, which means that the corresponding quantity $B^{(N)} = A_\varphi^{(N)}$ has been used for normalization of the coefficients in this ODE.

The second ODE for $w(z)$ (a counterpart of equation (3.14)) includes all the monomials from the initial relation excepting the monomial with index N. The coefficient of the nth-order derivative of $w(z)$ is equal to one; the coefficients of those monomials, for which $A_\varphi^{(i)} = 0$, are equal to $A^{(i)}$ and the coefficients $K_i(z)$ of the remaining monomials are to be determined.

The PDEs (counterparts of equations (3.13)), which determine functions of z and φ entering the similarity form, are constructed as follows

$$A^{(i)}(z,\varphi) = Q_i(z)A^{(N)}(z,\varphi) + K_i(z) \tag{3.15}$$

where i runs from 1 to M; $i \neq N$ and the values of i for which $A_\varphi^{(i)} = 0$ are also missing.

Remark (i). Some freedoms existing in the determination of the functions $Q_i(z)$ and $K_i(z)$ can be exploited to construct different explicit solutions of the system described above.

Remark (ii). A need to change the second normalizing coefficient (with index N) may arise while considering some special cases of the assumed similarity form, if for these cases this coefficient vanishes.

3.2. Example

To show the application of the method we will take a special case (A5) of the similarity form (1.8). The form (A5) corresponds to the same type of x and t dependence as most of the reductions obtained by the non-classical method including the new similarity reduction (2.23); applying the direct method due to Clarkson and Kruskal (1989) also leads to this form excepting the case $\beta =$ constant. We will show that by applying our method to the same simple form one can obtain additional similarity reductions which cannot be obtained by either the non-classical or the direct methods.

The coefficients $A^{(i)}$ specified for the form (A5) are given by (A6) (the rest of the coefficients vanish). Taking $N = 1$ in (3.15) and substituting (A6) into (3.15), we obtain the following equations:

$$-\frac{\beta'}{\gamma} - \frac{\gamma'\beta}{\gamma^2} = K_2 + Q_2\frac{\beta'}{\gamma} \tag{3.16}$$

$$-\frac{c}{\gamma}\left(\frac{\beta'}{\beta} + 2\frac{\gamma'}{\gamma}\right) = K_6 + Q_6\frac{\beta'}{\gamma} \tag{3.17}$$

$$-\frac{c^2\gamma'}{\beta\gamma^2} - \frac{\Theta}{\beta\gamma^3} = K_8 + Q_8\frac{\beta'}{\gamma}. \tag{3.18}$$

Here K_i and Q_i are constants since the functions β, γ and Θ depend only on t so that equations (3.16)–(3.18) have to be ordinary differential equations.

The system (3.16) and (3.17) of two equations for two variables $\gamma(t)$ and $\beta(t)$ can be easily solved by treating β as a new independent variable to give

$$\gamma = \gamma_0\beta^{-1/2}(\beta + a)^N \qquad (K_2 \neq 0,\, K_6 \neq 0) \tag{3.19a}$$

$$\gamma = \gamma_0\beta^{-Q_2-1} \qquad (K_2 = 0) \tag{3.19b}$$

$$\gamma = \gamma_0\beta^{-1/2}\exp(b\beta) \qquad (K_6 = 0) \tag{3.19c}$$

where

$$a = -\frac{2cK_2}{K_6} \qquad N = -Q_2 - \frac{1}{2} + \frac{K_2Q_6}{K_6} \qquad b = -\frac{Q_6}{2c} \tag{3.20}$$

and γ_0 is an arbitrary constant. The corresponding relations determining $\beta(t)$ are

$$\beta' = \frac{\gamma_0K_2\beta^{-1/2}(\beta + a)^{N+1}}{a[\beta(Q_6/2c) - (Q_2 + \frac{1}{2})]} \qquad (K_2 \neq 0,\, K_6 \neq 0) \tag{3.21a}$$

$$\beta' = \frac{\gamma_0K_6\beta^{-Q_2}}{c(2Q_2 + 1) - Q_6\beta} \qquad (K_2 = 0) \tag{3.21b}$$

$$\beta' = -\frac{\gamma_0K_2\beta^{-1/2}\exp(b\beta)}{Q_2 + \frac{1}{2} + b\beta} \qquad (K_6 = 0). \tag{3.21c}$$

After determination of the functions $\gamma(t)$ and $\beta(t)$, the function $\Theta(t)$ is explicitly defined by (3.18).

The overdetermined system of ODEs for the function $w(z)$ (it could also be obtained by a direct substitution of the relations (3.19) and (3.21) into equations (A6) and (A3)) has the form

$$w''' + K_2(w')^2 + K_6 w' + K_8 = 0 \tag{3.22}$$

$$ww'' + Q_2(w')^2 + Q_6 w' + Q_8 = 0. \tag{3.23}$$

We seek a solution of this system beginning with the second equation. By the reduction $w' = \pi(w)$ equations (3.22) and (3.23) are reduced to the following

$$(\pi\pi')'\pi + K_2\pi^2 + K_6\pi + K_8 = 0 \tag{3.24}$$

$$w\pi\pi' + Q_2\pi^2 + Q_6\pi + Q_8 = 0. \tag{3.25}$$

The second equation is easily integrated to give

$$(\pi + \lambda_1)^{\lambda_1} = Mw^S(\pi + \lambda_2)^{\lambda_2} \qquad (Q_2 \neq 0,\ Q_6 \neq 0,\ Q_8 \neq 0) \tag{3.26}$$

$$\pi - R_2 \ln|\pi + R_2| = \ln(Mw^{-Q_6}) \qquad (Q_2 = 0) \tag{3.27}$$

$$\pi = (-R_3 + Mw^{-2Q_2})^{1/2} \qquad (Q_6 = 0) \tag{3.28}$$

$$\pi = -R_1 + Mw^{-Q_2} \qquad (Q_8 = 0) \tag{3.29}$$

where

$$S = (Q_6^2 - 4Q_2Q_8)^{1/2} \qquad \lambda_1 = \frac{Q_6 - S}{2Q_2} \qquad \lambda_2 = \frac{Q_6 + S}{2Q_2}$$
$$R_1 = \frac{Q_6}{Q_2} \qquad R_2 = \frac{Q_8}{Q_6} \qquad R_3 = \frac{Q_8}{Q_2} \tag{3.30}$$

and M is an arbitrary constant. Substituting these formulae into (3.24) one can find all the cases for which the overdetermined system of equations (3.22) and (3.23) is compatible. After discarding variants which lead to solutions unbounded at infinity, we have three cases to consider.

Case 1. $K_2 = 0,\ K_8 = 0,\ Q_2 = -1,\ Q_8 = 0$:

$$\pi = Q_6 + Mw \qquad M^2 = -K_6 \tag{3.31}$$

where K_6 and Q_6 are arbitrary constants.

Case 2. $K_8 = 0,\ Q_2 = -2,\ Q_8 = 0$:

$$\pi = Q_6/2 + Mw^2 \qquad M = -K_2/6,\ K_6 = -Q_6K_2/3 \tag{3.32}$$

where K_2 and Q_6 are arbitrary constants.

Case 3. $K_6 = 0,\ K_8 = 0,\ Q_2 = -2,\ Q_6 = 0,\ Q_8 = 0$:

$$\pi = -(K_2/6)w^2 \tag{3.33}$$

where K_2 is an arbitrary constant.

The corresponding expressions for $w(z)$ are found by integrating the equation $w' = \pi(z)$.

Specifying the formulae (3.19)–(3.21) according to the values of constants from (3.31)–(3.33) and using in (A5) the resulting expressions for $\gamma(t)$, $\beta(t)$ and $w(z)$, we obtain the following explicit similarity solutions:

Case 1.

$$u = cz + \beta\left(\frac{R}{\lambda} + M\,\mathrm{e}^{-\lambda z}\right) \qquad z = x + q(t) \tag{3.34a}$$

$$R\beta + c\ln\beta = \lambda^2 t \qquad \Theta = 0. \tag{3.34b}$$

Case 2.

$$u = cz + \beta(6 \tanh z) \qquad z = x\frac{\beta}{t} + q(t) \tag{3.35a}$$

$$\beta = -\frac{c}{4} \pm \left(\frac{c^2}{16} + Mt^{2/3}\right)^{1/2} \qquad \Theta = c^2 t^{-3}(\beta^2 - t\beta\beta'). \tag{3.35b}$$

Case 3.

$$u = cz + t^{2/3}\left(\frac{6}{z}\right) \qquad z = xt^{-1/3} + q(t) \qquad \Theta = \frac{c^2}{3}t^{-5/3}. \tag{3.36}$$

In equations (3.34)–(3.36), c, M, R and λ are arbitrary constants and $q(t)$ is an arbitrary function.

All these solutions satisfy the condition (2.2) at infinity which takes in the variables (x, t, u) the form

$$u_x(x, t) = U^{(e)}(t) \qquad \text{as} \quad x \to \infty$$

where $U^{(e)}(t)$ is defined by the relation

$$U^{(e)}U_t^{(e)} = -\Theta(t).$$

The solution (3.36) is representative of the non-classical similarity reduction (2.23) with $w(z)$ being a particular solution of equation (2.23*b*) for $\Theta_0 = 0$. Two other solutions given by (3.34) and (3.35) correspond to new similarity reductions which do not figure among the reductions obtained by the non-classical method.

4. Concluding remarks

In this paper we have dealt with the problem of finding similarity reductions of the two-dimensional steady-state BL equations reduced to a single PDE (1.10) for the stream function of the flow. We have found several new similarity reductions and explicit solutions of this equation. Some of them are obtained by applying the non-classical group method due to Bluman and Cole (1969) and others are determined using an extension to the non-classical method introduced in this paper.

The extension is possible due to changes in the procedure of the method which are aimed at another representation of the resulting infinitesimal equation which expresses the invariance requirement for the initial PDE supplemented by the auxiliary equation(s) prescribing group invariance of the desired solutions. As distinct from the original procedure of the non-classical method, in which the resulting infinitesimal equation has the form of a polynomial in the derivatives of u with respect to x, in the changed procedure, this equation is represented as a polynomial in $w(z)$ and its derivatives. Two different ways of satisfying the infinitesimal equation lead to two different methods: the original non-classical method (or direct method) and the extension.

From another point of view such an approach provides a unifying group-theoretic framework for the direct method due to Clarkson and Kruskal (1989) and the method developed in Burde (1990, 1994). The group-theoretic explanation of the latter method, as an extension to the non-classical method, also defines an algorithm of its application.

We have given an example of applying the method to equation (1.10) to derive several explicit similarity solutions of this equation. Two of them, given by (3.34) and (3.35), correspond to the similarity reductions, which do not appear among either classical or non-classical reductions of the BL equations, and one, given by (3.36), is a representative of the non-classical similarity reduction with $w(z)$ being a particular solution of the corresponding ODE.

Note that this last solution might suggest that all the solutions obtained by our method do not represent independent similarity reductions but are only exact solutions corresponding to particular cases of more general similarity reductions. Therefore it is worth showing how the solution (3.36) can be a solution of the system of equations of our method and simultaneously satisfy the equations for the corresponding non-classical similarity reduction.

The solution (3.36) arises in our method if one specifies the values of constants according to (3.33) (the form of the solution also shows that one can set $K_2 = 1$ without loss of generality). Then $\gamma(t)$ and $\beta(t)$ determined by (3.19*c*) and (3.21*c*) take the forms corresponding to the non-classical similarity reduction (2.23*a*) which permits the reduction of the PDE (1.10) to one ODE (2.23*b*) for $w(z)$. However, in our method the function $w(z)$ is the solution of the overdetermined system of the ODEs (3.22) and (3.23) which, being specified according to the values of constants in (3.33) with $K_2 = 1$, take the following forms:

$$w''' + (w')^2 = 0 \qquad ww''' - 2(w')^2 = 0. \tag{4.1}$$

It is easily seen that multiplying the second equation of (4.1) by $\frac{2}{3}$ and adding the result to the first equation of (4.1) we obtain equation (2.23*b*) with $\Theta_0 = 0$. Thus, in this case dividing the relation (A3) in two, as in our method, is simultaneously dividing the ODE (2.23*b*) into two ODEs, which shows that the solution (3.36) is a peculiar case.

As regards two other solutions (3.34) and (3.35) obtained by our method, it can be checked that the corresponding expressions for $\gamma(t)$ and $\beta(t)$ do not allow (1.10) to be reduced to a single ODE but it is reduced to an overdetermined system which cannot be connected with a classical or non-classical reduction.

All the above considerations show that our method can produce similarity reductions which are not obtainable by either classical or non-classical group methods.

The solutions found with the help of our method—having rather simple forms—are nevertheless physically significant and can represent solutions of specific boundary-layer problems which are of interest from theoretical and engineering points of view—one can find examples relating to axisymmetric and unsteady flows in Burde (1994, 1995a). Examples of the exact solutions of the Navier–Stokes equations describing non-steady stagnation-point flows, which are important in many fields of aerodynamics and hydrodynamics and have numerous applications, are presented in Burde (1995b). The solutions (3.34)–(3.36) after specifying the free parameters contained in a solution (these are arbitrary constants and arbitrary functions of the longitudinal coordinate) can, for example, describe boundary-layer flows along solid permeable surfaces with a continuous distribution of suction (both normal and oblique). Transpiration at solid boundaries has numerous applications to boundary-layer control of flow over wings and turbine blades, the flow past permeable moving belt-surfaces with mass transfer found in industrial manufacturing devices and sundry chemical engineering processes. Asymptotic solutions to these problems obtained through similarity reductions of the BL equations help provide a fundamental understanding of these complicated flows.

The solutions found by our method can also be used as models for numerical experiments differing from known exact solutions of the BL equations in that they do not represent flows with self-similar velocity profiles. Therefore a specific problem corresponding to such a solution is not reduced to an ODE and the solution can serve as a test for essentially two-dimensional (or three-dimensional in the case of unsteady flows) numerical experiments.

Finally in this section, we remark that the similarity solutions of the steady-state two-dimensional BL equations, available with the help of our method, are not confined to the set found in this paper in which only an example of application of the method for a special case (A5) of the initial similarity form is presented. For instance, even a slight extension of the initial similarity form, namely

$$u = \alpha(x,t) + \beta(t)w(z) \qquad z = x\gamma(t) + q(t)$$

yields many more possibilities. How to get all the solutions of equation (1.10) obtainable by this method will be carefully discussed elsewhere.

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Appendix A. Application of the direct method to equation (1.10)

Even though it is known that the direct method due to Clarkson and Kruskal (1989) is equivalent to the non-classical method for the case $\tau \neq 0$ (see Olver 1994), we shall outline the application of the direct method to equation (1.10) since the corresponding formulae play an important role in the formulation of an extension to the non-classical method in section 3.

First, substituting the similarity form (1.7) into (1.10) yields

$$(F_w z_x^3)w''' + (3F_{ww} z_x^3)w'w'' + \cdots = 0. \tag{A1}$$

The rest of the terms have been omitted since applying the usual procedure of the direct method to these two is sufficient to reduce the similarity form to (1.8).

Substituting the form (1.8) into (1.10) and collecting coefficients of like derivatives and powers of $w(z)$ yields the relation

$$\begin{aligned}
&(\beta z_x^3)w''' + (\beta\beta_t z_x^2 - \beta\beta_x z_x z_t)ww'' + (\beta\beta_x z_x z_t + \beta^2 z_t z_{xx} - \beta\beta_t z_x^2 - \beta^2 z_x z_{xt})(w')^2 \\
&\quad +(\beta_t\beta_x z_x + \beta\beta_t z_{xx} + \beta\beta_{xx} z_t - \beta_x^2 z_t - \beta\beta_x z_{xt} - \beta\beta_{xt} z_x)ww' \\
&\quad +(\beta_t\beta_{xx} - \beta_x\beta xt)w^2 + (3\beta_x z_x^2 + 3\beta z_x z_{xx} + \alpha_t\beta z_x^2 - \alpha_x\beta z_x z_t)w'' \\
&\quad +(\beta z_{xxx} + 3\beta_x z_{xx} + 3\beta_{xx} z_x + \alpha_{xx}\beta z_t + 2\alpha_t\beta_x z_x + \alpha_t\beta z_{xx} \\
&\quad -\alpha_x\beta_x z_t - \alpha_x\beta_t z_x - \alpha_x\beta z_{xt} - \alpha_{xt}\beta z_x)w' \\
&\quad +(\beta_{xxx} + \alpha_t\beta_{xx} + \alpha_{xx}\beta_t - \alpha_x\beta_{xt} - \alpha_{xt}\beta_x)w \\
&\quad +(\alpha_{xxx} + \alpha_t\alpha_{xx} - \alpha_x\alpha_{xt} - \Theta) = 0.
\end{aligned} \tag{A2}$$

In order that this equation be an ordinary differential equation for $w(z)$, the ratios of coefficients of different derivatives and powers of $w(z)$ have to be functions of z only. This gives a set of equations for $\alpha(x,t)$, $\beta(x,t)$ and $z(x,t)$ whose solutions yield the similarity reductions of the original PDE (1.10). These equations are easier to examine if one makes the change of variables (3.8) in the coefficients. Then equation (A2) after dividing all the coefficients by the coefficient βz_x^3 of the monomial w''' (we assume that $z_x \neq 0$ which corresponds to the non-classical method for $\tau \neq 0$) may be rewritten in the form

$$\begin{aligned}
&\beta z_x^3[w''' + A^{(1)}ww'' + A^{(2)}(w')^2 + A^{(3)}ww' + A^{(4)}w^2 + A^{(5)}w'' + A^{(6)}w' + A^{(7)}w + A^{(8)}] \\
&\quad = 0
\end{aligned} \tag{A3}$$

where the normalized coefficients $A^{(i)}(z,\varphi)$ $(i = 1,\ldots,8)$ are expressed through the functions $\alpha(z,\varphi)$, $\beta(z,\varphi)$ and $\gamma(z,\varphi) = z_x$ by the relations

$$A^{(1)} = \left(\frac{\beta}{\gamma}\right) b_\varphi \qquad A^{(2)} = -\left(\frac{\beta}{\gamma}\right)(b_\varphi + g_\varphi) \tag{A4a}$$

$$\begin{aligned} A^{(3)} &= -\left(\frac{\beta}{\gamma}\right)(b_{z\varphi} + 2b_z g_\varphi - b_\varphi g_z) \\ A^{(4)} &= -\left(\frac{\beta}{\gamma}\right)(b_{z\varphi} b_z - b_{zz} b_\varphi + g_\varphi b_z^2 - b_\varphi b_z g_z) \end{aligned} \tag{A4b}$$

$$\begin{aligned} A^{(5)} &= 3(b_z + g_z) + \frac{1}{\gamma}\alpha_\varphi \\ A^{(6)} &= 3b_{zz} + 6b_z g_z + 3b_z^2 + g_{zz} + 2g_z^2 + \frac{1}{\gamma}(-\alpha_{z\varphi} + 2\alpha_\varphi b_z + \alpha_\varphi g_z - \alpha_z b_\varphi - 2\alpha_z g_\varphi) \end{aligned} \tag{A4c}$$

$$\begin{aligned} A^{(7)} = b_{zzz} &+ 3b_{zz} b_z + b_z^3 + 3b_{zz} g_z + b_z g_{zz} + 3b_z^2 g_z + 2g_z^2 b_z \\ &+ \frac{1}{\gamma}[-\alpha_{z\varphi} b_z + \alpha_{zz} b_\varphi + \alpha_\varphi(b_{zz} + b_z g_z + b_z^2) \\ &+ \alpha_z(-b_{z\varphi} - b_\varphi b_z + b_\varphi g_z - 2b_z g_\varphi)] \end{aligned} \tag{A4d}$$

$$A^{(8)} = \frac{1}{\beta}\left[\alpha_{zzz} + 3\alpha_{zz} g_z + \alpha_z g_{zz} + 2\alpha_z g_z^2 + \frac{1}{\gamma}(\alpha_\varphi \alpha_{zz} + \alpha_\varphi \alpha_z g_z - \alpha_z^2 g_\varphi - \alpha_z \alpha_{z\varphi}) - \frac{\Theta}{\gamma^3}\right] \tag{A4e}$$

with b and g defined in (3.9b).

Applying the procedure of the direct method due to Clarkson and Kruskal (1989) to (A3) in the case of $\beta_\varphi \neq 0$ yields the similarity reduction of the following form

$$u = cz + \beta(t) w(z) \qquad z = x\gamma(t) + q(t) \tag{A5}$$

where c is a constant. Then the non-vanishing normalized coefficients in (A3) are

$$\begin{aligned} &A^{(1)} = \frac{\beta'}{\gamma} \qquad A^{(2)} = -\frac{\beta'}{\gamma} - \frac{\gamma'\beta}{\gamma^2} \\ &A^{(6)} = -\frac{c}{\gamma}\left(\frac{\beta'}{\beta} + 2\frac{\gamma'}{\gamma}\right) \qquad A^{(8)} = -\frac{c^2\gamma'}{\beta\gamma^2} - \frac{\Theta}{\beta\gamma^3} \end{aligned} \tag{A6}$$

where β, γ and Θ are functions of t only. The constraints yielded by the requirement that (A3) with the coefficients in the forms (A6) is reduced to an ODE lead to the similarity reductions (2.23), (2.24) and (2.26).

The case $\beta_\varphi = 0$ yields the reduction (2.25).

Appendix B. Generalization of the method

The extension of the non-classical method, described in section 3, can be somewhat generalized if one does not assume $\tau = 1$ while deriving equation (3.7). Why τ does not remain arbitrary in the resulting infinitesimal equation of our procedure (as distinct from the original procedure of the non-classical method) is explained in detail in appendix C (to simplify the exposition there we take the Burgers equation as an initial PDE). Here we will only remark that even though the relations (3.2) (or (1.4) as in the original procedure of the non-classical method) determine the generators ξ and η only up to an overall functional multiple, the infinitesimal equation (1.3), in which these relations are used, does not possess such a property and therefore the resulting infinitesimal equation may also not have it. In other words, in spite of the fact that τ can be scaled out from (3.2) (or (1.4)) by redefining ξ and η as $\tilde{\xi} = \xi/\tau$ and $\tilde{\eta} = \eta/\tau$, the resulting infinitesimal equation may, along with $\tilde{\xi}$ and $\tilde{\eta}$, include terms with τ and its derivatives in such a way that one cannot take τ arbitrary (in particular, constant) without loss of generality. The difference between the resulting equation of the original non-classical method and ours is that such terms vanish in the original procedure and remain in ours.

Applying our procedure to the PDE (1.10) without assuming $\tau \neq$ constant (but it is assumed $\tau_u = 0$, for simplicity) yields the following resulting infinitesimal equation

$$P = \beta z_x^3[B^{(0)}w''' + B^{(1)}ww'' + B^{(2)}(w')^2 + B^{(3)}ww' + B^{(4)}w^2 + B^{(5)}w'' + B^{(6)}w' + B^{(7)}w + B^{(8)}] = 0 \tag{B1}$$

where the coefficients $B^{(i)}$ can be represented as follows

$$B^{(i)} = \tau A_{\varphi}^{(i)} - 3\tau_x(z_t/z_x)A^{(i)} \qquad i = 0, 1, \ldots, 8 \tag{B2}$$

with $A^{(0)}$=1. Thus, the resulting infinitesimal equation includes the additional terms having τ_x as a multiplier.

The generalization, like the method itself, stems from the idea that the resulting infinitesimal equation (B1) may be satisfied by reducing it to an ODE for $w(z)$ (the requirement that all the coefficients $B^{(i)}$ vanish leads, as before, to the direct method). This yields the following set of equations

$$\tau A_{\varphi}^{(m)} - 3\tau_x(z_t/z_x)A^{(m)} = k_m(z)[\tau(A^{(0)})_{\varphi} - 3\tau_x(z_t/z_x)A^{(0)}] \qquad m = 1, 2, \ldots, 8 \tag{B3}$$

where $k_m(z)$ are for the present arbitrary functions. Eliminating τ from the set of equations (B3) results in the system

$$\frac{(A^{(1)} - k_1A^{(0)})_{\varphi}}{A^{(1)} - k_1A^{(0)}} = \frac{(A^{(2)} - k_2A^{(0)})_{\varphi}}{A^{(2)} - k_2A^{(0)}} = \cdots = \frac{(A^{(8)} - k_8A^{(0)})_{\varphi}}{A^{(8)} - k_8A^{(0)}} \tag{B4}$$

which can be easily integrated to give

$$A^{(m)} - k_mA^{(0)} = \lambda_{ml}(z)(A^{(l)} - k_lA^{(0)}) \qquad m, l = 1, 2, \ldots, 8 \tag{B5}$$

where $\lambda_{ml}(z)$ are functions of integration.

Thus, the result is represented by the set of equations (B5), which leads to different variants of the method. If, for example, one selects from (B5) all the equations corresponding to some fixed $l = N$ (it is assumed that $A^{(N)} \neq 0$), the following system results

$$A^{(m)} = K_m(z)A^{(0)} + Q_m(z)A^{(N)} \qquad m = 1, 2, \ldots, 8 \ (m \neq N) \tag{B6}$$

where $K_m = k_m - \lambda_{mN}k_N$, $Q_m = \lambda_{mN}$ and $A^{(0)} = 1$. This corresponds to the method described above.

If one rearranges (B5) by composing the equations corresponding to two different values of the second subscript $l = N$ and $l = L$ ($A^{(N)} \neq 0$ and $A^{(L)} \neq 0$), this yields a system of the form

$$A^{(m)} = K_m(z)A^{(0)} + Q_m(z)A^{(N)} + R_m(z)A^{(L)} \qquad m = 1, 2, \ldots, 8 \ (m \neq N, L) \tag{B7}$$

where $K_m = k_m - (\lambda_{mN}k_N + \lambda_{mL}k_L)/2$, $Q_m = \lambda_{mN}/2$ and $R_m = \lambda_{mL}/2$. The system (B7) represents a generalization of the previous method when three of the coefficients $A^{(i)}$ (namely $A^{(0)} = 1$, $A^{(N)}$ and $A^{(L)}$) are used as normalizing coefficients. Other variants of the method can be derived from (B5) in a similar way.

Appendix C. Procedure in the case of $\tau \neq$ constant—an example of the Burgers equation

Here we will show how the terms including τ and its derivatives appear in the resulting infinitesimal equation of our procedure. We take the well known Burgers equation

$$\Delta = u_t + uu_x + u_{xx} = 0 \tag{C1}$$

as the initial PDE, in order not to deal with more complicated formulae for equation (1.10).

Then the infinitesimal equation (1.3) has the following form:

$$\begin{aligned} P = {} & u_{xt}u_x(-2\tau_u) + u_{xt}(-2\tau_x) + u_x^3(-\xi_{uu}) + u_x^2 u_t(-\tau_{uu}) + u_x^2(\eta_{uu} - 2\xi_{xu} + 2u\xi_u) \\ & + u_x u_t(2\xi_u - 2\tau_{xu}) + u_x(2\eta_{xu} - \xi_{xx} - \xi_t + u\xi_x + \eta) \\ & + u_t(2\xi_x - \tau_{xx} - \tau_t - u\tau_x) + (\eta_{xx} + \eta_t + u\eta_x) = 0. \end{aligned} \tag{C2}$$

Introducing new variables

$$\tilde{\xi} = \xi/\tau \qquad \tilde{\eta} = \eta/\tau \tag{C3}$$

one may rearrange equation (C2) as follows

$$\begin{aligned} P = {} & \tau\{u_x^3(-\tilde{\xi}_{uu}) + u_x^2(\tilde{\eta}_{uu} - 2\tilde{\xi}_{xu} + 2u\tilde{\xi}_u - 2\tilde{\xi}\tilde{\xi}_u) \\ & + u_x(2\tilde{\eta}_{xu} + \tilde{\eta} + 2\tilde{\eta}\tilde{\xi}_u - \tilde{\xi}_{xx} - \tilde{\xi}_t + u\tilde{\xi}_x - 2\tilde{\xi}\tilde{\xi}_x) \\ & + (\tilde{\eta}_{xx} + \tilde{\eta}_t + u\tilde{\eta}_x + 2\tilde{\eta}\tilde{\xi}_x)\} \\ & - S_1\{\tau_{xx} + \tau_{uu}u_x^2 + 2\tau_{xu}u_x + \tau_t - 2\tau_u\tilde{\xi}u_x + \tau_x(u - 2\tilde{\xi}) - 2\tau(\tilde{\xi}_u u_x + \tilde{\xi}_x)\} \\ & - S_2\{2\tau_x + 2\tau_u u_x\} = 0 \end{aligned} \tag{C4}$$

where

$$S_1 = u_t + \tilde{\xi}u_x - \tilde{\eta} \tag{C5}$$

$$S_2 = u_{xt} + u_x^2(\tilde{\xi}_u) - u_x(\tilde{\eta}_u - \tilde{\xi}_x + u\tilde{\xi} - \tilde{\xi}^2) - (\tilde{\eta}_x + \tilde{\xi}\tilde{\eta}). \tag{C6}$$

Here $S_1 = \Delta^{(1)}/\tau$, where $\Delta^{(1)}$ is the left-hand side of the invariant surface condition (1.4), and S_2 can be represented as follows:

$$S_2 = (S_1)_x + \tilde{\xi}(S_1 - \Delta). \tag{C7}$$

One can see that the terms in (C4) having S_1 and S_2 as multipliers do not permit taking τ arbitrary without loss of generality. The procedure of the non-classical method makes use of the equations

$$S_1 = 0 \qquad S_2 = 0 \tag{C8}$$

which removes those terms; as a result (C4) yields the determining equations of the non-classical method. Our procedure makes use of equations (3.2), (1.8) and (3.5), which satisfy the first equation of (C8) and its differential consequence $(S_1)_x = 0$, but it is not sufficient for satisfying the second equation of (C8). Therefore in the resulting infinitesimal equation obtained from (C4) with use of (3.2), (1.8) and (3.5):

$$\beta z_x^2[B^{(0)}w'' + B^{(1)}ww' + B^{(2)}w^2 + B^{(3)}w' + B^{(4)}w + B^{(5)}] = 0 \tag{C9}$$

the coefficients $B^{(i)}$ include additional terms proportional to τ_x (it is assumed $\tau_u = 0$):

$$B^{(i)} = \tau A_\varphi^{(i)} - 2\tau_x(z_t/z_x)A^{(i)} \qquad i = 0, 1, \ldots, 5. \tag{C10}$$

Here $A^{(i)}$ are the coefficients in the initial relation of the direct method, which is obtained by substituting the similarity form (1.8) into (C1) as follows

$$\beta z_x^2[A^{(0)}w'' + A^{(1)}ww' + A^{(2)}w^2 + A^{(3)}w' + A^{(4)}w + A^{(5)}] = 0 \tag{C11}$$

where $A^{(0)} = 1$.

The additional terms in (C10) appear since the procedure does not make use of equation (C11)—it is seen that substituting (C10) into (C9) and collecting the terms with τ_x will result in the form placed in the square brackets in (C11)—this is equivalent to retaining the term $-\tilde{\xi}\Delta$ in (C7) or the corresponding term $2\tau_x\tilde{\xi}\Delta$ in (C4). Although it looks somewhat artificial, the fact that the resulting infinitesimal equation includes those terms does not violate its validity whereas it permits a generalization of the method.

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